3D Ising Model on Dual BCC Lattice: the Sign-Factor

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Abstract

The 3d Ising model on a regular cubic lattice can be expressed in terms of an $SU(2)$ 2d fermionic model with a Z_2-fluxes. We modify the model such that it is defined on the dual to a body centered cubic lattice. The advantage of this lattice is that 2d embedded surfaces have no selfintersections, thus partially avoiding the Sign-factor problem associated with the 2d fermionoc models related to the 3d Ising model. Rather than solving the full $SU(2)$ fermionic theory on this lattice we study the simpler model of scalar fermions and find the spectrum of excitations. The model has no mass gap. We reformulate the model using the R-formalism and a new interesting structure appears due to the necessity of introducing a three-paticle matrix $R^{(3)}_{ijk}$. It encodes the essential character of the Sign-factor. We analyse the integrability properties of this class of models.

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1 Introduction

After a conjecture by A.Polyakov [1] that the three-dimensional Ising Model (3DIM) can be represented as a fermionic string theory in a three dimensional Euclidean space a number of works was carried out during the eighties [2, 3, 4, 5, 6, 7, 8, 9, 10] trying to substantiate the conjecture. The main problem is the so called Sign-factor, which is the analogy of the Kac-Ward factor [11] for the Ising Model formulated on a 2D regular lattice. This Sign-factor ensures the cancellation of self-intersecting 2D surfaces mapped onto the target space 3D regular lattice. The cancellation is necessary in order to have correct Boltzmann weights in the string representation of the 3D statistical sum as an ensemble of the random 2D surfaces [2, 3, 6, 7, 8]. It resembles the Pauli principle for Fermi particles, which prohibits two of them to be in the same state (in the same space-time point in this case). Therefore it is expected that the Sign-factor is connected with some fermionic structure on the 2d world-sheet of the strings.

In [8] a fermionic model with the Sign-factor was defined on the 2D regular lattice, and its continuum limit was investigated in [9, 12]. The model appeared to be a model of SO(3) fermions hopping in the staggered field of Z_2-fluxes on a random Manhattan lattice.

A similar model of fermions hopping in a staggered $U(1)$-gauge field background on a regular Manhattan lattice was analysed in [12].

In [15] a model of spins with Z_2 global symmetry was considered on the so called dual Body Centered Cubic (BCC) lattice (see Fig.1). The BCC lattice consists of two simple cubic sublattices arranged in such a way that the sites of one sublattice are positioned in the middle points of the cubes of the second sublattice (see Fig.1a).

The dual to the BCC (DBCC) lattice is represented in the Fig.1b. As one can see, the two dimensional faces of the dual BCC lattice are:
i) hexagons, which are dual to the links connecting the neighbour vertices of different sublattices (dotted lines in the Fig.1b) and
ii) squares, which are orthogonal to the links of the same sublattice (bold lines in the Fig.1b).

This lattice is interesting in a sense that the relevant surfaces here, i..e. those which at most occupy the faces of the DBCC lattice once, have no self-intersections at all. The analogy for contours on a 2D lattice is the honeycomb (or its dual triangular) lattice, where curves have no self-intersection points.

We follow the same reasoning as in [8] in the case of a regular lattice, but now on a DBCC lattice. We then obtain a fermionic hopping model in a staggered $SU(2)$-gauge field, defined on the Manhattan lattice of the type shown in Fig. 3. On this lattice we do not have the magnetic fluxes which represented the ends of the self-intersection lines on the cubic lattices. However, we will see the emergence of a new structure of the action, which encodes the essential character of the Sign-factor and which is interesting from the point of view of integrability of the model.

In this article we define and investigate a model of fermions hopping in a staggered $U(1)$-gauge field background on the ML defined in Fig.3. We find the spectrum of the model and analyse its continuum limit. We introduce a three particle scattering R^3-matrix, which, together with the ordinary two particle R^2-matrix constitutes the Monodromy matrix of the model by appearing alternating in a product. We analyse the integrability property of this type of model and write down the corresponding Yang-Baxter equations (YBE).

2 The Model

We consider now a simple 2d fermionic system hopping on the ML (Fig.3) which was constructed on particular surfaces on the DBCC lattice as mentioned above. This simplification will allow us to analyse the continuum limit of the model and investigate integrability structure.

The ML structure defines the hopping of fermions only along the arrows of the lattice (see Fig. 3) and the Hamiltonian is non hermitian:

$$H = \sum_{<\vec{n},\vec{m}>} t_{\vec{n},\vec{m}} c^{+}(\vec{n}) U_{\vec{n},\vec{m}} c(\vec{m}) + \sum_{\vec{n}} c^{+}(\vec{n}) c(\vec{n}). \tag{1}$$

Here $U_{\vec{n},\vec{m}}$ are the group elements of the external $U(1)$ field defined according to demand that the magnetic fluxes around the all plaquettes with circulating arrows will be $\phi = 2\pi p/q$ (p and q are mutually prime integer numbers). The hopping parameters $t_{\vec{n},\vec{m}}$ are chosen to be periodic, reflecting the translational symmetry of the lattice by vectors $2\vec{i}_x$, $2(1+\sqrt{3})\vec{i}_y$,where $\vec{i}_x$ and $\vec{i}_y$ are unit vectors in coordinates direction. It allows us to distinguish 10 types of particles (Fig.1):

In order to diagonalize the Hamiltonian (1) we pass to the Bloch wave basis:

$$\begin{aligned} c_i(\vec{k}) &= \frac{1}{\sqrt{2\pi N_x N_y}} \sum_{n_x,n_y} e^{-i\vec{k}(\vec{n}+\vec{r}_i)} c_i(\vec{n}), \\ c_i^{+}(\vec{k}) &= \frac{1}{\sqrt{2\pi N_x N_y}} \sum_{n_x,n_y} e^{i\vec{k}(\vec{n}+\vec{r}_i)} c_i^{+}(\vec{n}), \\ k_{x(y)} &= \frac{2\pi n_{x(y)}}{N_{x(y)}}, \quad n_{x(y)} = 0, ..., N_{x(y)}-1. \end{aligned}$$

($N_x, 2N_y$ are the numbers of the hexagons in the corresponding directions), and the Hamiltonian becomes:

$$H(t_{i.j}, \Phi) = \sum_{\vec{k}} c_i^{+}(\vec{k}) H_{i,j}(\vec{k}, \Phi) c_i(\vec{n}), \tag{2}$$

where $H(k_x, k_y)$ is a 10×10 matrix:

$$\begin{pmatrix} 1 & t_{12}e^{ip} & 0 & t_{14}e^{iq} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{21}e^{-ip} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{20}e^{\frac{-iQ}{2}} \\ 0 & t_{32}e^{-iq} & 1 & t_{34}e^{-ip} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{43}e^{-ip} & 1 & t_{45}e^{iQ} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{53}e^{\frac{iP}{2}} & 0 & 1 & t_{56}e^{iQ} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & t_{67}e^{-ip} & 0 & t_{69}e^{iq} & 0 \\ 0 & 0 & 0 & 0 & t_{75}e^{\frac{iP}{2}} & t_{76}e^{-ip} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{87}e^{-iq} & 1 & t_{89}e^{ip} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{98}e^{ip} & 1 & t_{90}e^{\frac{-iP}{2}} \\ t_{01}e^{\frac{-iP}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & t_{08}e^{\frac{-iQ}{2}} & 0 & 1 \end{pmatrix}$$

We have defined $k_x = p$, $\sqrt{3}k_y = q$, $p + q = Q$ and $p - q = P$.

The partition function for the excitations of the states with energy E is the functional integral over the Grassmann variables $\psi_i(\vec{n}), \bar{\psi}_i(\vec{n})$ corresponding to the fermions.

$$Z(E) = tre^{-\beta H} = \int \prod d\psi_i(\vec{n}) d\bar{\psi}_i(\vec{n}) e^{-H(\psi,\bar{\psi},\Phi)+E\bar{\psi}\psi}. \tag{3}$$

Since this integral is Gaussian with respect to $\psi_i(\vec{n}), \bar{\psi}_i(\vec{n})$ fields it follows that

$$Z(E) = \prod_{k_x,k_y} \left[\det H(k_x, k_y) - E\right]. \tag{4}$$

The equation $\det H(k_x, k_y) = 0$ gives the critical line for the low lying excitations.

The investigation of the excitations is conveniently done by passing from the Lagrangian (action) formalism in the $(1+1)$ Euclidean space-time (formula 3) to the Hamiltonian and consider the representation of the partition function via the Transfer Matrix as follows

$$Z(0) = tr[T]^{N_y}. \tag{5}$$

By analysing the periodic structure of the ML under consideration, the complete Transfer Matrix can be written as the product of four transfer matrices T_i, $i = 1, 2, 3, 4$ each of which mediates the evolution between the states $|t+i-1\rangle$ and $|t+i\rangle$, respectively, (Fig. 3):

$$|\, t + i\rangle = T_i \mid t + i - 1\rangle \tag{6}$$

and the complete Transfer Matrix is equal to

$$T = T_1 T_2 T_3 T_4. \tag{7}$$

The technique of passing from the Lagrangian to the Hamiltonian for models on ML was developed in [12], based on the concept of coherent states [14]. It is possible to introduce two types of coherent states which are the eigenstates of the creation or annihilation operators, respectively. As shown in [12] for a ML, we need to consider two types of coherent sates alternating in a chain. One state differs from other by particle-hole transformation. Consequently we will need a particle and hole ordering prescription in order to complete the definition of the Transfer Matrices. Let us consider particle $|\psi\rangle$ and hole $|\bar{\psi}\rangle$ states as follows

$$\begin{aligned}
|\psi_{2i}\rangle &= e^{\psi^{2i}c^+_{2i}}|0\rangle, & \langle\bar{\psi}_{2i}| = \langle 0|e^{c_{2i}\bar{\psi}_{2i}}, \\
|\bar{\psi}_{2i+1}\rangle &= (c^+_{2i+1} - \bar{\psi}_{2i+1})|0\rangle, & \langle\psi_{2i+1}| = \langle 0|(c_{2i+1} + \psi_{2i+1}). \\
\langle\bar{\psi}_{2i}|\psi_{2i}\rangle &= e^{\bar{\psi}_{2i}\psi_{2i}}, & \langle\psi_{2i+1}|\bar{\psi}_{2i+1}\rangle = e^{\psi_{2i+1}\bar{\psi}_{2i+1}}
\end{aligned} \tag{8}$$

with the properties

$$\begin{aligned}
\int d\bar{\psi}_{2i}d\psi_{2i}|\psi_2 i\rangle\langle\bar{\psi}_2|e^{\bar{\psi}_{2i}\psi_{2i}} = 1, \\
\int d\bar{\psi}_{2i+1}d\psi_{2i+1}|\bar{\psi}_{2i+1}\rangle\langle\psi_{2i+1}|e^{\bar{\psi}_{2i+1}\psi_{2i+1}} = 1.
\end{aligned} \tag{9}$$

As it is clear from Fig. 3, the Transfer Matrices T_2 and T_4 are permuting the disposition of particle and hole type of coherent states at the odd and even sites, and because of this the definition of the coherent state of the whole chain at the Euclidean times $|t{+}2\rangle$, $|t{+}3\rangle$ must be interchanged.

Following [12] it is easy to see that the Transfer Matrices T_i are of the form :

$$T_i =: \exp -H_i :, \tag{10}$$

$$\begin{aligned}
H_1 &= \sum_i [-t_{12}c^+_1(2i-1)c_2(2i) + t_{34}c^+_2(2i)c_1(2i-1) - \\
&- (t_{14}+1)c^+_1(2i-1)c_1(2i-1) + (t_{32}+1)c^+_2(2i)c_2(2i)],
\end{aligned} \tag{11}$$

$$\begin{aligned}
H_2 &= \sum_i [-t_{43}c^+_1(2i+1)c_2(2i) + t_{45}t_{53}c^+_1(2i-1)c_2(2i) + t_{76}c^+_2(2i+1)c_1(2i) - \\
&- t_{65}t_{57}c^+_2(2i-1)c_1(2i) - (1-t_{56}t_{45})c^+_1(2i)c_1(2i-1) + \\
&+ (1-t_{75}t_{53})c^+_2(2i)c_2(2i-1)],
\end{aligned} \tag{12}$$

$$\begin{aligned}
H_3 &= \sum_i [-t_{67}c^+_1(2i)c_2(2i-1) + t_{89}c^+_2(2i-1)c_1(2i) + \\
&- (t_{69}+1)c^+_1(2i)c_1(2i) + (t_{87}+1)c^+_2(2i-1)c_2(2i-1)],
\end{aligned} \tag{13}$$

$$\begin{aligned}
H_4 &= \sum_i [-t_{98}c^+_1(2i)c_2(2i+1) + t_{90}t_{08}c^+_1(2i)c_2(2i-1) + t_{21}c^+_2(2i)c_1(2i+1) - \\
&- t_{20}t_{01}c^+_2(2i)c_1(2i-1) + (1-t_{90}t_{0,1})c^+_1(2i)c_1(2i-1) + \\
&+ (1-t_{20}t_{08})c^+_2(2i)c_2(2i-1)],
\end{aligned} \tag{14}$$

where under the simbol :: we mean ordinary normal ordering of the fermionic operators at the even sites of the chain and anti-normal ordering for the odd sites.

Evaluating now the product $T_1T_2T_3T_4$ for the whole Transfer Matrix T (7) in the basis of coherent states and puting it into the formula (5), one can obtain the expression (3) for the partition function Z.

Making a Fourier transformation of the Hamiltonians (10) and expressing the product (7) of the Transfer matrices by use of the generators of the sl_2 algebra in a Schwinger form

$$
\begin{aligned}
T &= De^{-H} = D\prod_p e^{-H_p}, && (15)\\
H_p &= \vec{\varepsilon}\vec{S}_p + \mu(n_{p1}+n_{p2}), && (16)\\
S_1 &= c_1^+c_2 - c_2^+c_1, \quad S_2 = c_1^+c_2 + c_2^+c_1, && (17)\\
S_3 &= c_1^+c_1 - c_2^+c_2, \quad S_0 = I, && (18)
\end{aligned}
$$

one can, after some algebra, obtain the spectrum of the excitations. In order to do that, and for simplicity, from now on we will consider two particular types of parameterization of the hopping amplitudes:

i)

$$
\begin{aligned}
t_{12} &= t_{14} = t_{34} = t_{32} = t_{67} = t_{69} = t_{87} = t_{76} = t_{89} = t_{98} = t_{21} = t_{43} = t,\\
t_{53} &= t_{56} = t_{45} = t_{75} = t_{08} = t_{90} = t_{20} = t_{01} = t^{\frac{1}{2}}
\end{aligned} \tag{19}
$$

and

ii)

$$
\begin{aligned}
&t_{i5} = t_{5j} = t_{i0} = t_{0j} = t,\\
&t_{ij} = t_1, \quad i,j \neq 5,0, \quad t_1/t = x.
\end{aligned} \tag{20}
$$

The first parameterization is chosen by simplicity arguments while the second is demanded by the rotation invariance.

First we will analyse the case i). The eigenvalues of the Hamiltonian (15) is found to be $E = \pm\varepsilon(p)$ with the following dispersion relation, chemical potential μ and the parameter D

$$
\begin{aligned}
\cosh\varepsilon(p) &= 2\sin^2\tfrac{\Phi}{2}\left(\left(1+4\cos^2\Phi\right)t^4 + 4\cos\Phi t^2 - 2\cos\Phi\right) + 2 + 1/2t^{-4} + 2\cos\Phi t^{-2}\\
&\quad +2\cos 2p\left(2\sin^2\tfrac{\Phi}{2}\left((2\cos\Phi-1)t^2 - 2\cos\Phi t^4 + 1\right) - 2\cos\Phi - t^{-2}\right)\\
&\quad +\cos 4p\left(1 - 4\sin^2\tfrac{\Phi}{2}t^2\right),\\
\mu &= 0,\\
D &= t^{4N_x}.
\end{aligned} \tag{21}
$$

The condition when the massless excitations appear is defined by the equation

$$
\begin{aligned}
&2\sin^2\tfrac{\Phi}{2}\left((1-8\sin^2\tfrac{\Phi}{2}\cos\Phi)t^8 + (8\cos\Phi-4)t^6 + 4\sin^2\tfrac{\Phi}{2}t^4\right) + \tfrac{1}{2} =\\
&= 2t^4(2\cos\Phi - 1) + 4\sin^2\tfrac{\Phi}{2}t^2.
\end{aligned} \tag{22}
$$

The limit $t \to \infty, \quad \Phi \to 0$ on the critical line (22) becomes

$$\sin \frac{\Phi}{2} t^2 = 1, \tag{23}$$

and the dispersion relation simplifies to

$$\cosh \varepsilon = 12(1 - \cos 2p) + \cos 4p. \tag{24}$$

We see that at the point p=0 there is no gap in the spectrum $\varepsilon(0) = 0$, and the excitations energy near that point is a linear function of the momentum:

$$\varepsilon(p) = \pm 4\sqrt{2} p. \tag{25}$$

In the case *ii*) the model under discussion here is connected with the Chalker-Coddington phenomenological model [12, 13] for the edge excitations in Hall effect responsible for the plateau-plateau transitions.

The equation of the spectrum in this case is the following

$$\begin{aligned}
&\cosh \varepsilon(p) = 1 + (\cos 4p - 1)(t^{-2} + 2(\cos\Phi - x^2)t) + \\
&2(\cos 2p - 1)\left(xt^3 \cos\Phi(2x^2\cos\Phi - 1 - x^4) + \right. \\
&\left. -2t^2x^{-1}\cos\Phi(\cos\Phi - x^2) + x(x^2 - \cos\Phi) - t^{-1}x{-}1\cos\Phi - t^{-4}x^{-1}\right),
\end{aligned} \tag{26}$$

and the equation of the critical line takes the form

$$\begin{aligned}
&(x^4 + 1 - 2x^2\cos\Phi)x^2\left(2t^{10}\cos\Phi - 2x\cos\Phi t^9 + x^2t^8/2\right) \\
&+(x^2 - \cos\Phi)\left(4t^8x\cos\Phi - 2t^7x^2 - t^6 + 2t^5x\right) + (1 - x^2\cos\Phi)\left(4t^7x^2\cos\Phi - 2t^6x^3\right) \\
&+(\cos\Phi^2 - x^2)t^6 = 2t^5x\cos\Phi - t^4x^2 + 2t^2x.
\end{aligned} \tag{27}$$

It is easy to see from this equation that for the fully packed phase: $t_{i,j} \to \infty$, there are two possible choices of the background U(1) field's phase: $\cos\Phi = x^2 = \pm 1$. In that limit eq. (27) becomes

$$(1 - \cos\Phi) = \frac{1}{4t^5}. \tag{28}$$

In both cases we find that the spectrum

$$\cosh \varepsilon(p) = 1 - 2(\cos 2p - 1)\frac{1}{t} + (\cos 4p + 4\cos 2p - 5)\frac{1}{t^2} \tag{29}$$

when $t \to \infty$ becomes $\varepsilon(p) = 0$ for the all values of the momentum.

3 Representation of the partition function Z via two- and three-particle R-matrices

In this section we will demonstrate that the partition function (3) can be constructed as a trace of the N-th power (N is a size of the lattice in a time direction) of some transfer matrix T

$$Z = trT^N, \tag{30}$$

but unlike the case of ordinary integrable models, the transfer matrix T here is a product of four transfer matrices presented in the formulas (7), (10) and (11). However, if we look at the Fig.3 under a 45° angle and use the corresponding time direction, we see only two different rows of products of constituent R-matrices (see Fig.4), corresponding to following Monodromy matrices in the so called braid formalism

$$T_0(u) \;=\; \prod_{j=0}^{L} R^{(2)}_{3j-1,3j}(u) R^{(3)}_{3j,3j+1.3j+2}(u), \tag{31}$$

$$T_1(u) \;=\; \prod_{j=0}^{L} R^{(3)}_{3j,3j+1,3j+2}(u) R^{(2)}_{3j+2,3j+3}(u). \tag{32}$$

Here the $R^{(2)}_{i,i+1}(u)$'s are the ordinary two-particle scattering R-matrices corresponding to the squares of the chain in the Fig. 4, and the $R^{(3)}_{i-1,i,i+1}(u)$'s correspond to the hexagons and represent three-particle scattering R-matrices. Therefore the Monodromy matrix of the model is the product $T(u) = T_1(u)T_0(u)$.

The two-particle R_{ij}-matrix is an operator acting on the direct product of the two two dimensional spaces (spaces of the fermions with 0 spin) defined on the sites i and j, which according to the technique developed in [16] can be fermionised by considering the basis states as $|k\rangle$, $k = 0, 1$, with $|1\rangle = c^+|0\rangle$:

$$R^{(2)}_{ij}|i_1\rangle \otimes |j_1\rangle = (R_{ij})^{i_2 j_2}_{i_1 j_1}|i_2\rangle \otimes |j_2\rangle. \tag{33}$$

Graphically it is represented in Fig.5b, in the same way as in [17] for the R-matrices of the XXZ-model in the braid formalism,

The three-particle R_{ijk}-matrix is an endomorphism on the direct product of three two-dimensional spaces (see Fig.5a) with a basis defined as before:

$$R^{(3)}_{ijk}|i_1\rangle \otimes |j_1\rangle \otimes |k_1\rangle = (R_{ijk})^{i_2 j_2 k_2}_{i_1 j_1 k_1}|i_2\rangle \otimes |j_2\rangle \otimes |k_2\rangle. \tag{34}$$

Following to the [16] we can represent $R^{(s)}$-matrices ($s = 2, 3$) via fermionic creation-annihilation operators by considering the Hubbard operators

$$X^i_j = |j\rangle\langle i|, \qquad i, j = 0, 1, \tag{35}$$

and taking into account that the fermionic Fock space is graded with the parities of the states defined as $p(i) = i$. Then, by definition

$$
\begin{aligned}
R^{(2)}_{ij} &= R_{ij}|i_1\rangle|j_1\rangle\langle j_1|\langle i_1| = (R_{ij})^{i_2 j_2}_{i_1 j_1}|i_2\rangle|j_2\rangle\langle j_1|\langle i_1| \\
&= (-1)^{p(i_1)(p(j_1)+p(j_2))}(R_{ij})^{i_2 j_2}_{i_1 j_1} X^{i_1}_{i_2} X^{j_1}_{j_2},
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
R^{(3)}_{ijk} &= R_{ijk}|i_1\rangle|j_1\rangle|k_1\rangle\langle k_1|\langle j_1|\langle i_1| \\
&= (R_{ijk})^{i_2 j_2 k_2}_{i_1 j_1 k_1}|i_2\rangle|j_2\rangle|k_2\rangle\langle k_1|\langle j_1|\langle i_1| \\
&= (-1)^{p(i_1)(p(j_2)+p(j_1))+(p(i_1)+p(j_1))(p(k_1)+p(k_2))}(R_{ijk})^{i_2 j_2 k_2}_{i_1 j_1 k_1} X^{i_1}_{i_2} X^{j_1}_{j_2} X^{k_1}_{k_2}.
\end{aligned}
\tag{37}
$$

The fermionic expression of $R^{(2)}$ for the XXZ model can be found in [16](and references there), while the most general form of the three-particle scattering $R^{(3)}_{123}$ operator is

$$
\begin{aligned}
R^{(3)}_{123} = \; & R^{000}_{000} n_1 n_2 n_3 + R^{001}_{001} n_1 n_2 \bar{n}_3 + R^{010}_{010} n_1 \bar{n}_2 n_3 + R^{100}_{100} \bar{n}_1 n_2 n_3 + R^{011}_{011} n_1 \bar{n}_2 \bar{n}_3 \\
& + R^{101}_{101} \bar{n}_1 n_2 \bar{n}_3 + R^{110}_{110} \bar{n}_1 \bar{n}_2 n_3 + R^{111}_{111} \bar{n}_1 \bar{n}_2 \bar{n}_3 + (R^{010}_{001} c_2^+ c_3 + R^{001}_{010} c_3^+ c_2) n_1 \\
& + (R^{110}_{101} c_2^+ c_3 + R^{101}_{110} c_3^+ c_2) \bar{n}_1 + (R^{100}_{001} c_1^+ c_3 + R^{001}_{100} c_3^+ c_1) n_2 + (R^{110}_{011} c_1^+ c_3 + \\
& R^{011}_{110} c_3^+ c_1) \bar{n}_2 + (R^{100}_{010} c_1^+ c_2 + R^{010}_{100} c_2^+ c_1) n_3 + (R^{101}_{011} c_1^+ c_2 + R^{011}_{101} c_2^+ c_1) \bar{n}_3,
\end{aligned}
\tag{38}
$$

Where $n_i = c_i^+ c_i$ and $\bar{n}_i = 1 - n_i$. The two-particle $R^{(2)}_{ij}$-matrix of the XXZ model can be obtained from (38) by putting $\bar{n}_3 = 1$ everywhere and taking $(R^{(2)})^{i_2 j_2}_{i_1 j_1} = (R^{(3)})^{i_2 j_2 1}_{i_1 j_1 1}$ and $R^{i_2 j_2 0}_{i_1 j_1 0} = 0$ otherwise.

Both kinds of R-matrices can be written as an exponent

$$
:\exp c_i^+ A_{ij} c_j:,
\tag{39}
$$

where the indices i, j run from 1 to 2 for $R^{(2)}_{12}$ operators and to 3 for the three-particle $R^{(3)}_{123}$ operator. The notion $:\;\;:$ means the normal ordering for the c_i^+ , c_i (i=1,3) and the hole ordering for the c_2^+ , c_2 operators.

The connection between the matrix elements $A_{i,j}$ and the hopping parameters $t_{i,j}$ in the action of the model can be found by the relation

$$
\langle\bar{\psi}_1|\langle\psi_2|\langle\bar{\psi}_3| :\exp c_i^+ A_{ij} c_j: |\psi_3\rangle|\bar{\psi}_2\rangle|\psi_1\rangle = \exp(-\bar{\psi}_i t_{i,j} \psi_j).
\tag{40}
$$

and are

$$
A_{ij} = \begin{cases} -t_{ij}, & i \neq j, \quad i = 1,3 \\ t_{ij} & \qquad\quad\; i = 2 \end{cases}, A_{ii} = \begin{cases} 1 + t_{ii} & i = 2 \\ -(1 + t_{ii}) & i = 1,3 \end{cases}.
\tag{41}
$$

In (40) the coherent states $|\psi_i\rangle$ are defined by the formulas (8).

The matrix elements $(R^{(3)})_{ijk}^{lmn}$ in the expression (38) are connected with the A_{ij}-s in (39) by the following equations

$$\begin{array}{lll} R_{101}^{101} = 1, & R_{101}^{110} = A_{23}, & R_{110}^{101} = A_{32}, \\ R_{011}^{011} = (1+A_{11})(1-A_{22}) + A_{12}A_{21}, & R_{011}^{101} = A_{12}, & R_{101}^{011} = A_{21}, \\ R_{110}^{110} = (1+A_{33})(1-A_{22}) + A_{23}A_{32}, & \multicolumn{2}{l}{R_{011}^{110} = A_{13}(1-A_{22}) + A_{12}A_{23},} \\ R_{111}^{111} = 1 - A_{22}, & \multicolumn{2}{l}{R_{110}^{011} = A_{31}(1-A_{22}) + A_{21}A_{32},} \end{array} \tag{42}$$

$$\begin{array}{ll} R_{001}^{010} = A_{23}(1+A_{11}) - A_{13}A_{21}, & R_{010}^{001} = A_{32}(1+A_{11}) - A_{31}A_{12}, \\ R_{001}^{100} = -A_{13} & R_{100}^{001} = -A_{31}, \\ R_{010}^{100} = A_{12}(1+A_{33}) - A_{32}A_{13}, & R_{100}^{010} = A_{21}(1+A_{33}) - A_{23}A_{31}, \end{array} \tag{43}$$

$$\begin{array}{l} R_{001}^{001} = 1 + A_{11}, \\ R_{010}^{010} = (1-A_{22})(1+A_{11}+A_{33}) - \det A + A_{11}A_{33} - A_{13}A_{31} + A_{12}A_{21} + A_{23}A_{32}, \\ R_{100}^{100} = 1 + A_{33}, \\ R_{000}^{000} = (1+A_{11})(1+A_{33}) - A_{13}A_{31}. \end{array} \tag{44}$$

The two-particle $R_{ij}^{(2)}$-matrix can be obtained from these expressions by taking $A_{i3} = A_{3j} = 0$ everywhere.

This is the general form of three-particle $R_{ijk}^{(3)}$ matrix, but for the model under consideration (3) we should consider free fermionic limit and take $A_{22} = 1$ and $A_{13} = A_{13} = 0$.

The condition of integrability of this model with two- and three-particle R-matrices, namely the condition of commutativity of Transfer matrices (31) for different values of the spectral parameter, can be written as a modified Yang-Baxter equation in the following form

$$\mathbf{R}_{12}(u,v)R_{234}(u)R_{45}(u)R_{12}(v)R_{234}(v) = R_{234}(v)R_{45}(v)R_{12}(u)R_{234}(u)\mathbf{R}_{45}(u,v), \tag{45}$$

were $\mathbf{R}_{ij}(u,v)$ is the intertwiner operator. This equations definitely has a solution in free fermionic case corresponding to the model defined above, since we were able to diagonalise the Hamiltonian. It would be interesting to find a solution for a general case of $R^{(3)}$ matrix (42-44) and $R^{(2)}$-matrix of the XXZ model.

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Figure captions

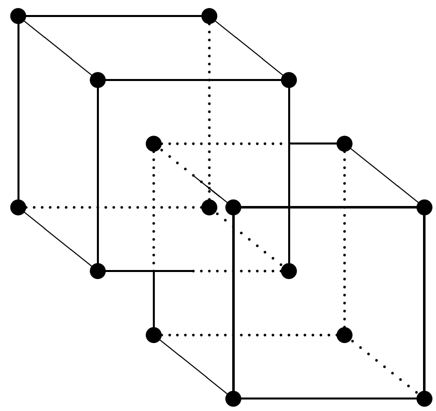
1a. The BCC lattice
1b. The Dual BCC lattice

2. How the Manhattan Lattice is forming on surfaces on Dual BCC lattice

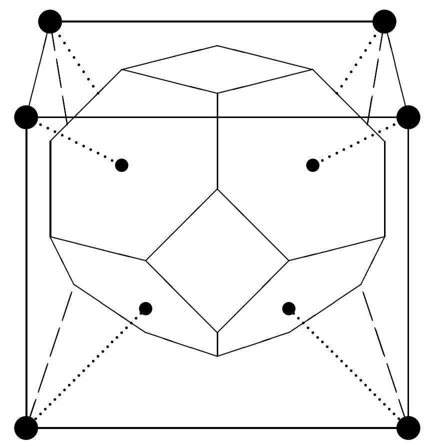
3. Manhattan lattice appeared on 2D surfaces on Dual BCC lattice

4. Two rows of Monodromy matrices

5a. Graphical representation of $R^{(2)}_{ij}$ matrix
5b. Graphical representation of $R^{(3)}_{ijk}$ matrix



a)



b)

Fig.1

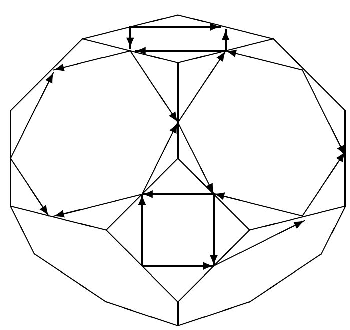


Fig.2

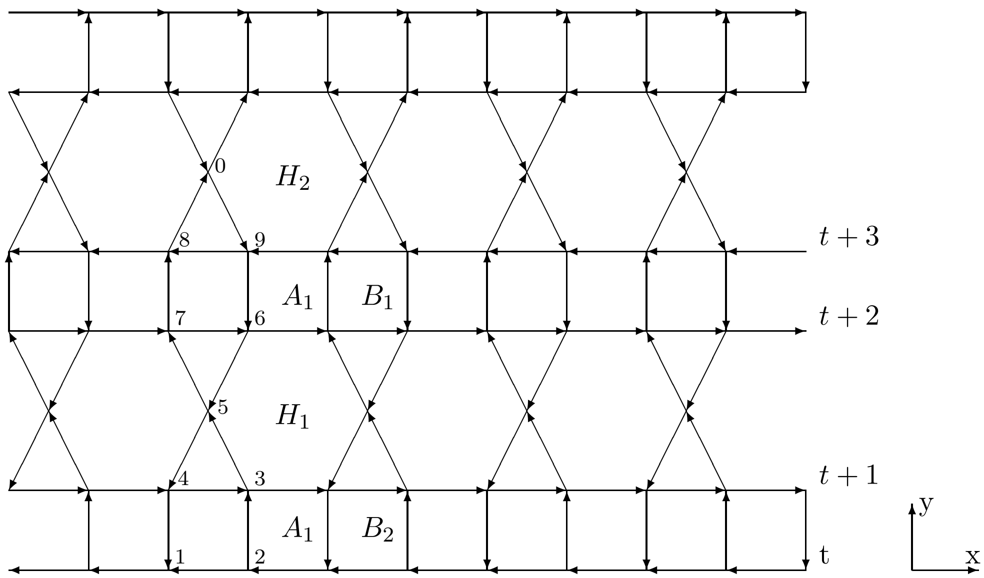


Fig.3

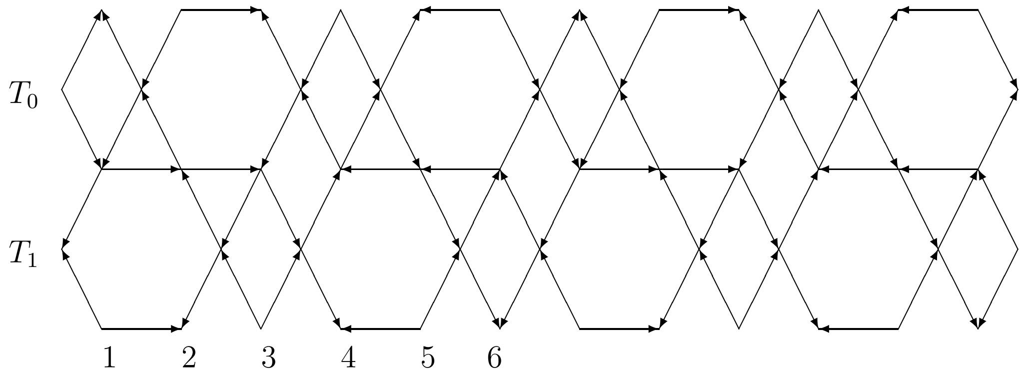


Fig.4

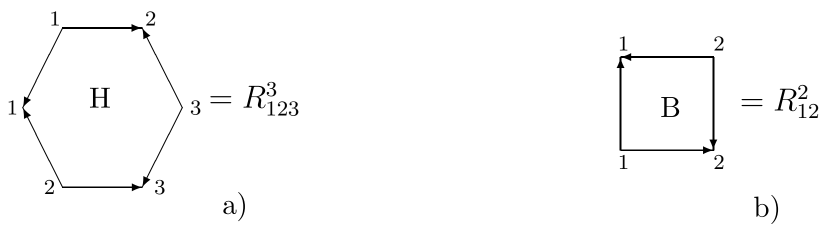


Fig.5